Contextualidade Quântica: o que é e onde pode nos ajudar

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Plan

- Nonlocality
- Contextuality
- Both together
- Current/Future Works

Nonlocality

Historical approach

- Measurement reveals preassigned values
- Freedom to choose among allowed measurements
- Choices/results at one part do not influence on choices/results at another

Physical Background

- Extrinsic Probabilities: they come from ignorance
- Determinism
- No spooky action at a distance

Mathematical Formulation

Joint conditional probabilities (correlations) $p\left(a,b|x,y\right)$

Parts Two (for simplicity)

Outcomes $a\in\mathcal{A}, b\in\mathcal{B}$

Measurements $x\in\mathcal{X}, y\in\mathcal{Y}$

$\mathcal{A},\mathcal{B},\mathcal{X},\mathcal{Y}$ Finite sets

Mathematical Formulation

Joint conditional probabilities (correlations) $p\left(a,b|x,y\right)$

Locality:

$$p\left(a,b|x,y\right)=\sum_{\lambda}p\left(\lambda\right)\ p\left(a|x,\lambda\right)\ p\left(b|y,\lambda\right)$$

Examples

$$a, b, x, y \in \{0, 1\}$$

$$p(a, b|x, y) = \frac{1}{4} \qquad \forall a, b, x, y$$

$$p(a|x) = \frac{1}{2} = p(b|y)$$

Examples

$$a, b, x, y \in \{0, 1\}$$

$$p\left(0, 0|x, y\right) = 1$$

$$p\left(0|x\right) = 1 = p\left(0|y\right)$$

Examples

$$a, b, x, y \in \{0, 1\}$$

$$p(a, b|x, y) = p(a, b)$$

$$p(a, b) = \sum_{\lambda=(a,b)} p(\lambda)\,\delta(a|\lambda)\,\delta(b|\lambda)$$

Examples

$$a, b, x, y \in \{0, 1\}$$

$$p(0,0|0,0) = p(1,1|0,0) = \frac{1}{2}$$

$$p(0,1|0,1) = p(1,0|0,1) = \frac{1}{2}$$

$$p(0,1|1,0) = p(1,0|1,0) = \frac{1}{2}$$

$$p(0,0|1,1) = p(1,1|1,1) = \frac{1}{2}$$

Exercise!

Bell Inequalities

Locality assumption implies restrict bounds for sums of joint conditional probabilities

Making a long story short:

Quantum Theory allows for violations of Bell inequalities

Example

$$S = p\,(00|00) + p\,(11|00) + p\,(00|01) + p\,(11|01) +$$
$$+\, p\,(00|10) + p\,(00|10) + p\,(01|11) + p\,(10|11)$$

Clearly

$$S \leq 4$$

With locality

$$S \leq 3$$

Clauser, Horne, Shimony, Holt, *PRL* (1969)

Non-Local Examples

$$a, b, x, y \in \{0, 1\}$$

$$p\,(0, 0|0, 0) = 1$$
$$p\,(0, 0|0, 1) = 1$$
$$p\,(0, 0|1, 0) = 1$$
$$p\,(1, 1|1, 1) = 1$$

Exercise 2

Non-Local Examples

$$a, b, x, y \in \{0, 1\}$$

$$p(0,0|0,0) = p(1,1|0,0) = \frac{1}{2}$$
$$p(0,0|0,1) = p(1,1|0,1) = \frac{1}{2}$$
$$p(0,0|1,0) = p(1,1|1,0) = \frac{1}{2}$$
$$p(1,0|1,1) = p(0,1|1,1) = \frac{1}{2}$$

Exercise 3

Popescu, Rohrlich, *Found. Phys.* (1994)

What is the difference?

$$p(a\,|\,x) = \,?$$

$$p\,(0,0|0,0) = 1$$
$$p\,(0,0|0,1) = 1$$
$$p\,(0,0|1,0) = 1$$
$$p\,(1,1|1,1) = 1$$

Exercise 2

$$p\,(0,0|0,0) = p\,(1,1|0,0) = \frac{1}{2}$$
$$p\,(0,0|0,1) = p\,(1,1|0,1) = \frac{1}{2}$$
$$p\,(0,0|1,0) = p\,(1,1|1,0) = \frac{1}{2}$$
$$p\,(1,0|1,1) = p\,(0,1|1,1) = \frac{1}{2}$$

Deterministic

Signalling

Non-Deterministic Exercise 3

Non-Signalling

Popescu, Rohrlich, *Found. Phys.* (1994)

Geometrical Perspective

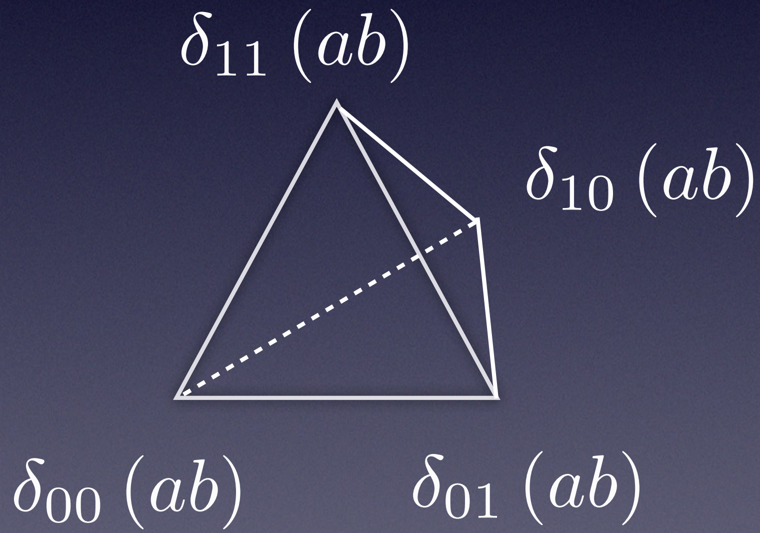
From the beginning

$$p(a) = p(a=0)\,\delta_0(a) + p(a=1)\,\delta_1(a)$$

$\delta_0(a)$ $\delta_1(a)$

From the beginning

$$p(a,b) = \sum_{\alpha,\beta} p(a=\alpha, b=\beta)\ \delta_{\alpha,\beta}(a,b)$$

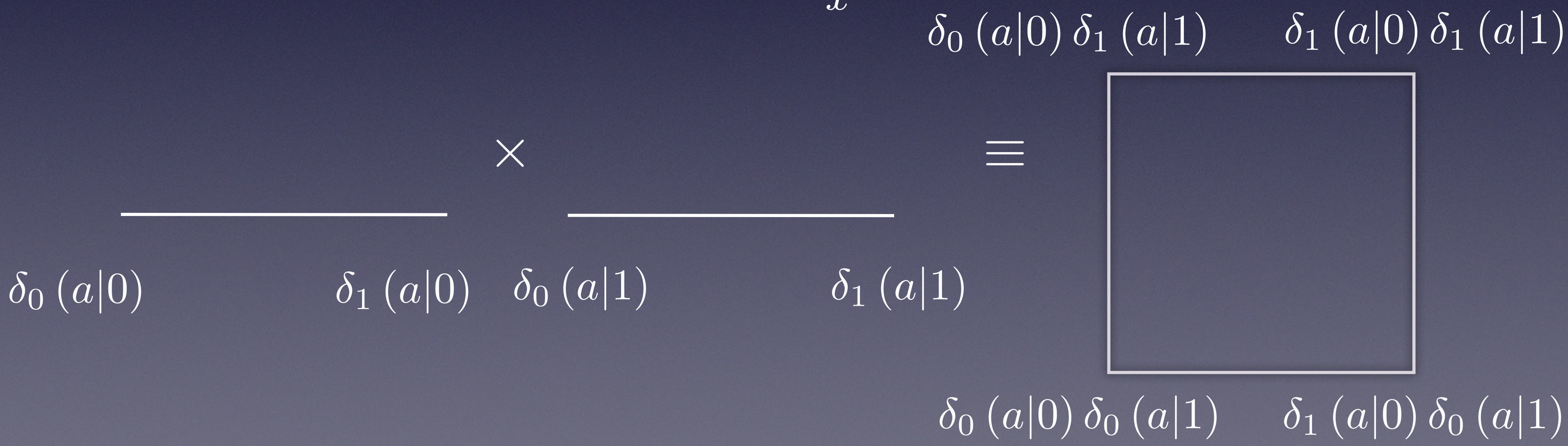


Probability Simplex

For Conditional Probabilities

$$p\left(a|x\right) = \sum_{\alpha} p\left(\alpha|x\right)\ \delta_{\alpha}\left(a|x\right)$$

$$\mathcal{P}_{\alpha|x} = \prod_{x} \mathcal{P}_{\alpha}$$



Polytope

Correlation Polytope

Almost a tautology

$$p\left(a,b|x,y\right)=\sum_{\alpha\beta}p\left(\alpha,\beta|x,y\right)\ \delta_{\alpha\beta}\left(a,b|x,y\right)$$

$\delta_{\alpha\beta}\left(a,b|x,y\right)$ Deterministic probabilities
Extremal points

Meaning: any "correlation" is a <u>convex combination</u> of those deterministic distributions

Bell Inequalities

Bounds on sums of "correlations":

Hyperplanes defining Hyperspaces!

Finitely many

Polytope: intersections of finitely many hyperspaces

Vertices and Facets

Dual descriptions of polytopes

Tight inequalities: those saturated on facets

Other interesting sets

Quantum Set

$$p\,(a,b|x,y) = \mathrm{tr}\left(\rho_{AB}\ M_{a|x} \otimes N_{b|y}\right)$$

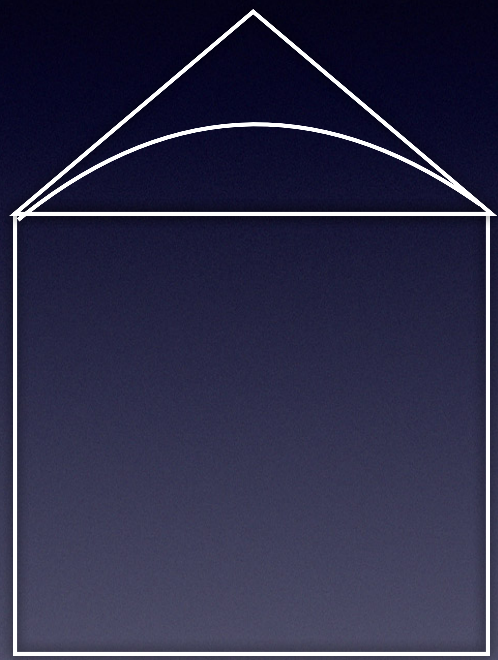
Not a polytope!
A convex set

Non-Signalling Set

$$p(a|x,y) := \sum_b p(a,b|x,y) = p(a|x)$$

$$p(b|x,y) := \sum_a p(a,b|x,y) = p(b|y)$$

A Cartoon



$$\mathscr{L} \subset \mathscr{Q} \subset \mathscr{NS}$$

Usefulness

- Ekert 91 protocol for QKD
- One can mimic Nonlocality using communication
  - Nonlocality is a resource!
    - Secrecy
    - Randomness

Finally… Contextuality

Nonlocality

Historical approach

- Measurement reveals preassigned values
- Freedom do choose among allowed measurements
- Choices/results at one part do not influence on choices/results at another

Contextuality

Historical approach

- Measurement reveals preassigned values
- Freedom do choose among allowed measurements
- Choices of compatible measurements do not influence on choices/results of another

Physical Background

For locality

- Extrinsic Probabilities: they come from ignorance
- Determinism
- No spooky action at a distance

Physical Background

For noncontextuality

- Extrinsic Probabilities: they come from ignorance
- Determinism
- Things are what they are: noncontextuality

Mathematical Formulation

Joint conditional probabilities (correlations) $p\left(a,b|x,y\right)$

Parts: Two (for simplicity)

Outcomes: $a \in \mathcal{A}, b \in \mathcal{B}$

Measurements: $x \in \mathcal{X}, y \in \mathcal{Y}$

$\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}$: Finite sets

Mathematical Formulation

Joint conditional probabilities (correlations) $p\left(\{a_i\}|\{x_i\}\right)$

Measurements $x_i \in \mathcal{X}$

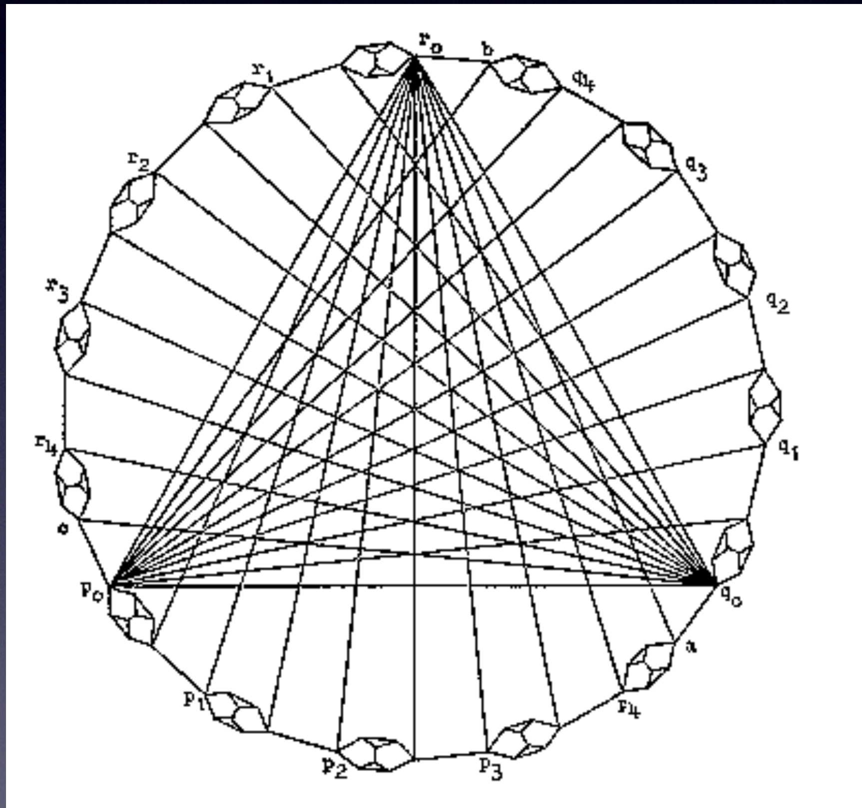
Outcomes $a_i \in \mathcal{A}_i$

Contexts $\mathfrak{c} : i, j \in \mathfrak{c} \Rightarrow x_i, x_j$ compatible

Compatibility Graphs

First Historical Example

State Independent Proof



Thm: There is no non-contextual assignalation consistent with quantum mechanics

Kochen, Specker, *J. Math. Mech.* (1967)

Simplest Example

$$n = 18, \quad d = 4$$

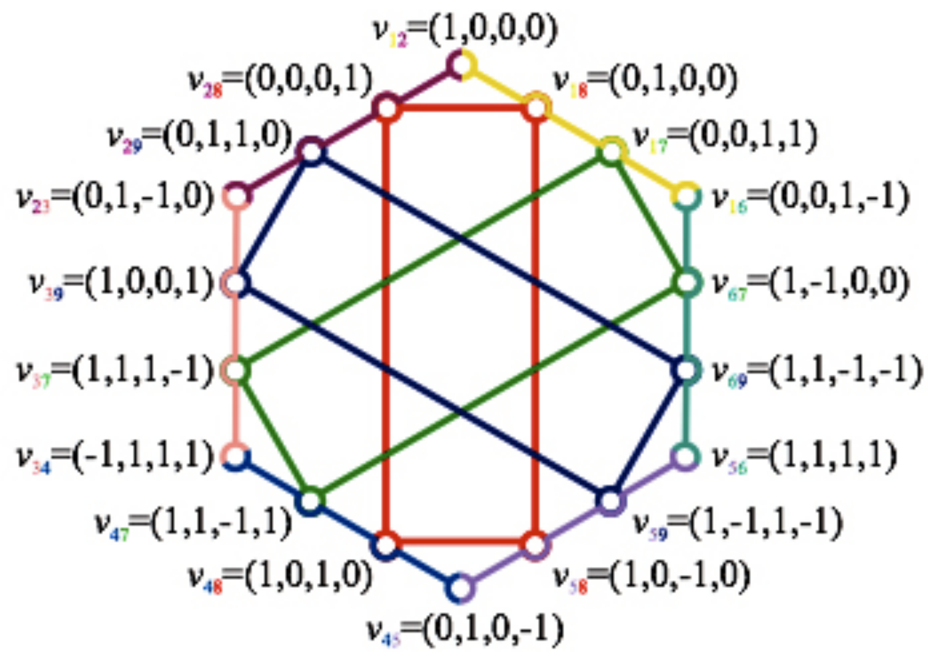


Figure 1.2: Vectors for the 18-projector proof of the Kochen-Specker theorem.

Parity Proof

Cabello, Estebaranz, Garcia Alcaine, *Phys Lett. A* (1996)

Even Simpler

Peres-Mermin Square

$\sigma_x \otimes I$	$I \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$
$I \otimes \sigma_z$	$\sigma_z \otimes I$	$\sigma_z \otimes \sigma_z$
$\sigma_x \otimes \sigma_z$	$\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$

Peres, *Phys. Lett. A* (1990); Mermin, *Phys. Rev. Lett.* (1990)

Mathematical Formulation

Joint conditional probabilities (correlations) $p\left(\{a_i\}|\{x_i\}\right)$

$$p\left(\{a_i\}|\{x_i\}\right) = \sum_{\lambda} p\left(\lambda\right) \prod_{x_i \in \{x_i\}} p\left(a_i|x_i, \lambda\right)$$

Boole Inequalities

Noncontextuality assumption implies restrict bounds for sums of joint conditional probabilities

And once more...

Quantum Theory allows for violations of Boole inequalities

Geometrical Perspective

Boole Inequalities

Bounds on sums of "correlations":

Hyperplanes defining Hyperspaces!

Finitely many

Polytope: intersections of finitely many hyperspaces

Vertices and Facets

Dual descriptions of polytopes

Tight inequalities: those saturated on facets

Other interesting sets

Quantum Set

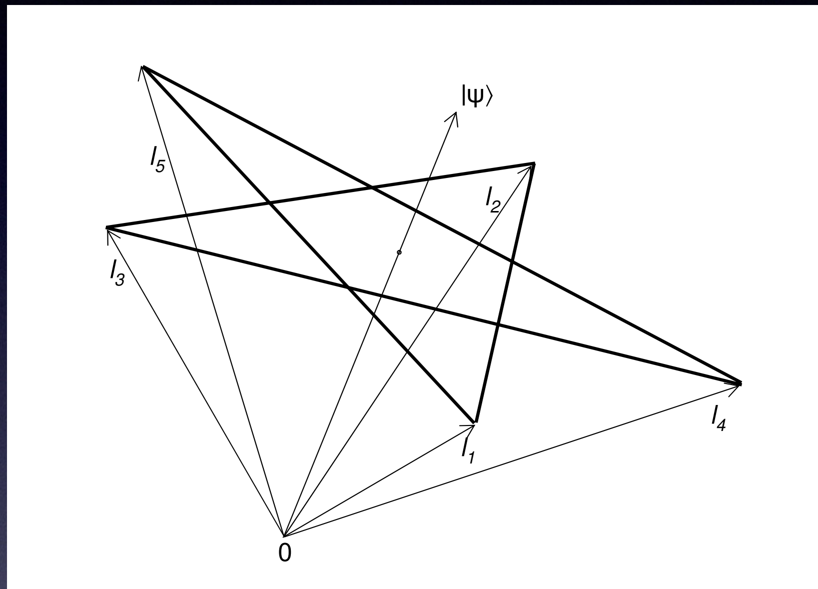
$$p\left(\{a_i\}|\mathfrak{c}\right) = \mathrm{tr}\left(\rho \prod_{i\in\mathfrak{c}} P_{a_i|i}\right)$$

Not a polytope!
A convex set

Nondisturbing Set

$$p\left(a_i|i\right) = p\left(a_i|\mathfrak{c}\right) := \sum_{j\in\mathfrak{c},j\neq i} p\left(\{a_i\}|\mathfrak{c}\right)$$

Example



$$p(01|01) + p(01|12) + p(01|23) + p(01|34) + p(01|40) \leq 2$$

Klyachko, Can, Binicioglu, Shumovsky, *PRL* (2008).

Conceptual Remark

- State Independent Contextuality
  - Kochen-Specker original proof
  - Cabello 1996
  - Peres-Mermin Square…
- State Dependent Contextuality
  - KCBS inequality
  - n-Cycle inequalities
  - Nonlocality

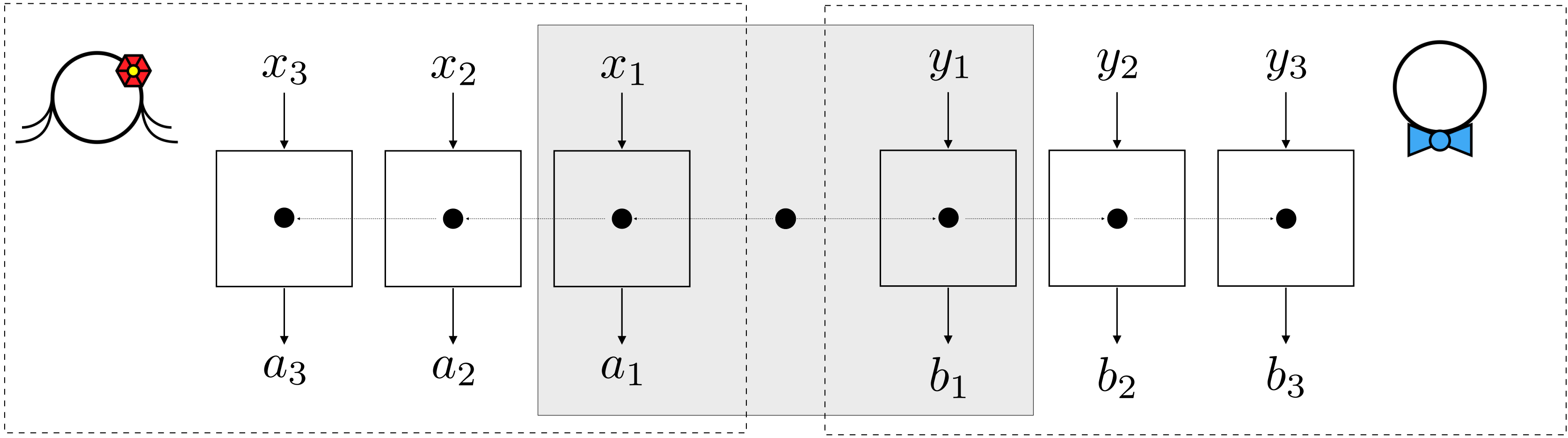
Usefulness

- Magic State Distillation
- One can mimic Contextuality using memory
  - Contextuality is a resource!
    - Processing
    - Randomness
    - Nonclassicality Certification

Both together

(If time allows)

Generalised Bell Scenarios



$$|\Psi\rangle = \left(\cos\frac{\pi}{8}\,|\psi_-\rangle + \sin\frac{\pi}{8}\,|\phi_+\rangle\right)^{\otimes 2}$$

$\sigma_x \otimes I$	$I \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$
$I \otimes \sigma_z$	$\sigma_z \otimes I$	$\sigma_z \otimes \sigma_z$
$\sigma_x \otimes \sigma_z$	$\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$

$\sigma_x \otimes I$	$I \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$
$I \otimes \sigma_z$	$\sigma_z \otimes I$	$\sigma_z \otimes \sigma_z$
$\sigma_x \otimes \sigma_z$	$\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$

2 PM

Locality Definition

Usual

$$p(a, b \,|\, x, y) = \sum_{\lambda} p(\lambda)\; p(a \,|\, x, \lambda)\; p(b \,|\, y, \lambda)$$

New

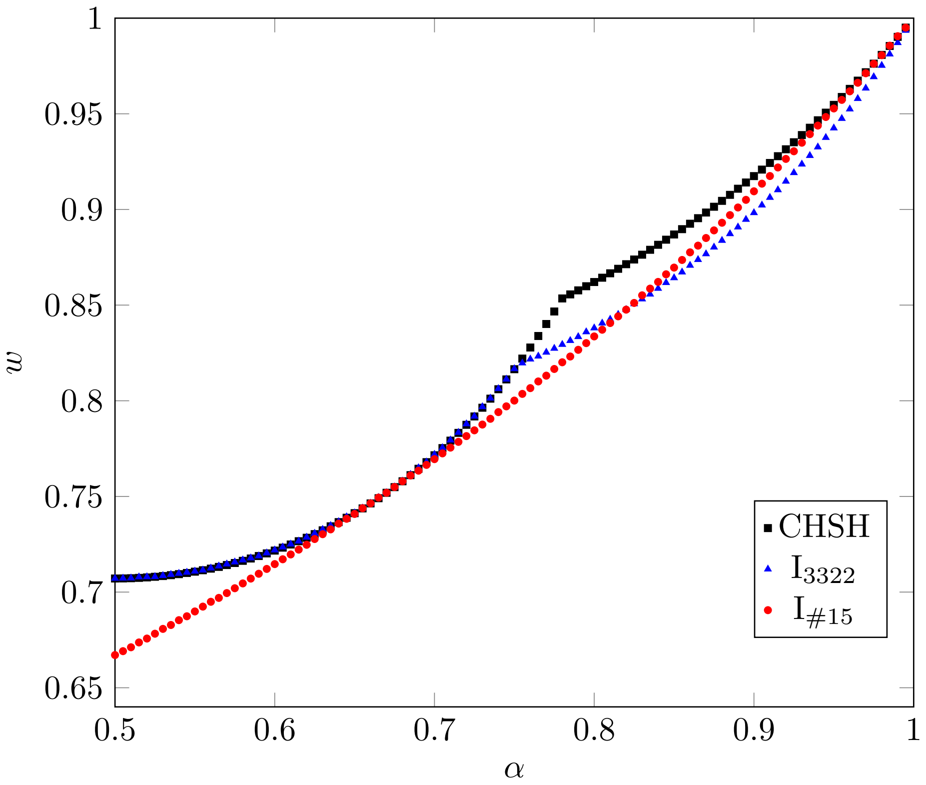
$$p(\mathrm{a}, \mathrm{b} \,|\, \mathrm{x}, \mathrm{y}) = \sum_{\lambda} p(\lambda)\; p(\mathrm{a} \,|\, \mathrm{x}, \lambda)\; p(\mathrm{b} \,|\, \mathrm{y}, \lambda)$$

Suitably/Wisely chosen $p(\mathrm{a} \,|\, \mathrm{x}, \lambda),\ p(\mathrm{b} \,|\, \mathrm{y}, \lambda)$

Temistocles, Rabelo, Terra Cunha, *Phys. Rev. A* (2019)

Xiao, Ruffolo, Mazzari, Temistocles, Terra Cunha, Rabelo, Xue *arXiv 2204.05385*

More Nonlocal States

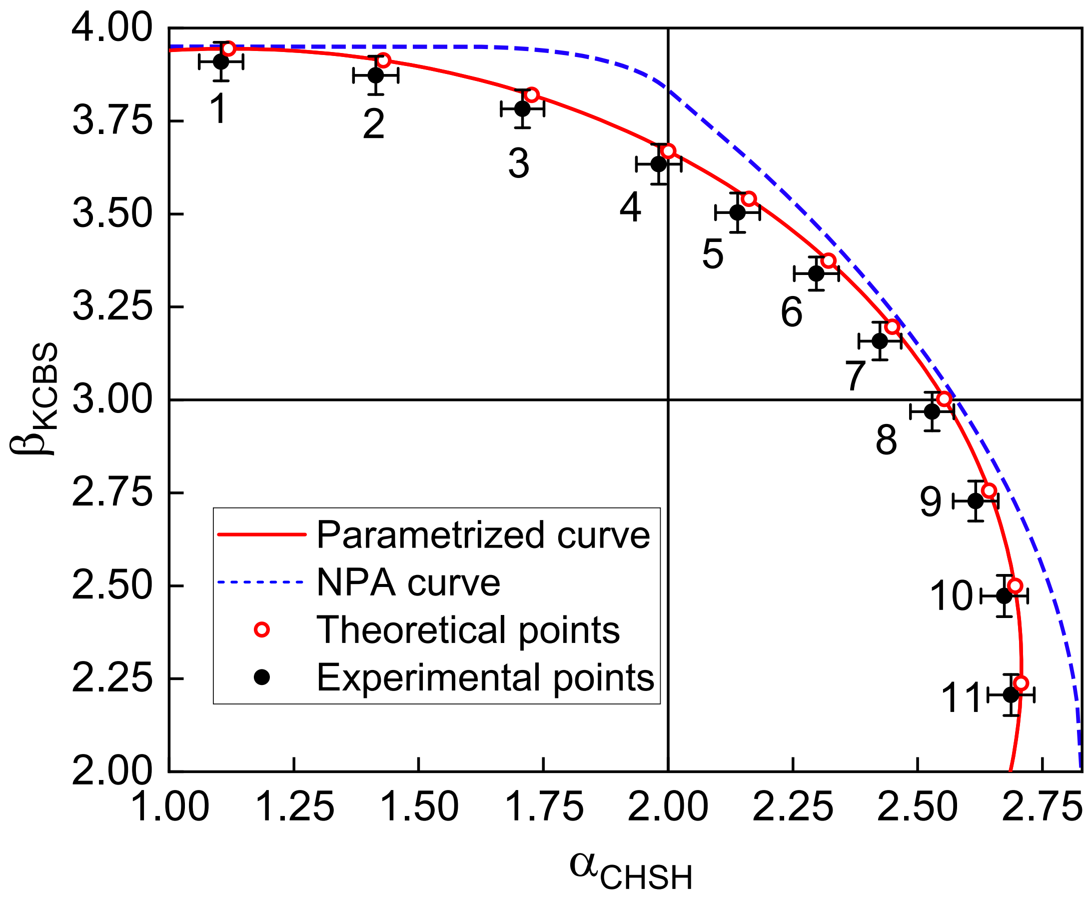


$$\rho(\alpha, w) = w \left|\psi(\alpha)\right\rangle \left\langle\psi(\alpha)\right| + (1-w)\left|00\right\rangle\left\langle 00\right|$$

$$\left|\psi(\alpha)\right\rangle = \sqrt{\alpha}\left|01\right\rangle + \sqrt{1-\alpha}\left|10\right\rangle$$

Temistocles, Rabelo, Terra Cunha, *Phys. Rev. A* (2019)

Nonlocality & Contextuality



Xiao, Ruffolo, Mazzari, Temistocles, Terra Cunha, Rabelo, Xue *arXiv 2204.05385*

Thank you!

Estado Democrático de Direito Sempre!